Indian Statistical Institute, Bangalore
M. Math. II Year, First Semester
Mid-Sem Examination
Fourier Analysis

Time: 3 hours September 27, 2010

Maximum marks you can get is 40. Part A with 20 marks is compulsory. So max marks you can get in parts B is 20.

Notation For any function f, the fourier transform of f is denoted by $\widehat{f}$

PART - A

1. Let $f_1(x) = (1-x^2)e^{-\frac{x^2}{2}}$.
   Show that $\int dt \quad \frac{|\widehat{f_1}(t)|^2}{|t|} < \infty$. [2]

2. Let $g\varepsilon L^2(R)$ be given by $\widehat{g}(t) = \chi^{(|t|)}_{[\pi,\ 2\pi]} \quad e^{it/2}$. Show that the family $\{g(2^p t - r) : p, r \text{ are integers }\}$ is a complete family in the Hilbert space $L^2(R)$. [4]

3. Let $SL^2(n,j)$, be the L^2 spline functions of order n on standard intervals of length 2^{-j}, given by $SL^2(n,j) = \{f\varepsilon L^2(R) : f\varepsilon C^{n-1}(R)$, and on $[r2^{-j}, (r+1)2^{-j}]$ is a polynomial of degree $\le n$ for each integer $r\}$.
   Here j is an integer. Show that $\bigcap\limits_{j=-\infty}^{\infty} SL^2(n,j) = 0$. [3]

4. Calculate the Fourier transform of the following functions.
   a) $f_1(t) = e^{-t}$ for $t \ge 0, 0$ for $t \le 0$
   b) $f_2(t) = e^t$ for $t \le 0, 0$ for $t > 0$
   c) $e^{-|t|}$
   d) Let $f(t) = f_1(t) - f_2(t)$. Find $limit_{\delta \longrightarrow 0}$,
   $\widehat{[f(\delta .)]} \quad (\xi)$
   e) Give $g\varepsilon L'(R)$ such that $\widehat{g}$ is not in $L'(R)$ and prove your claim. [5]

5. a) Show that
   $\sup\limits_{|y| \le a} \quad \int\limits_{|x|\ge 2a} |\frac{1}{x-y} - \frac{1}{x}|\ dx \le \frac{k}{2a}$. for some constant k, independent of $a > 0$. [3]

6. Let $\psi \varepsilon L^2(R) \cap L'(R), f \varepsilon L^2(R)$. Define $\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\ \psi[\frac{t-b}{a}]$. For $a > 0, b$ real. Show that $\int dbe^{-itb} < f, \psi_{a,b} >$

   $= k\ \sqrt{a}\ \widehat{f}(t)\ \overline{\widehat{\psi}(at)}$

   for some constant k independent of f. [3]

PART - B

7. Let $f_2 \in L' \cap L^2(R)$, *supp* f_2 bounded and $\int f_2 = 0$. Show that

   $\int dt \quad \frac{|\widehat{f_2}(t)|^2}{|t|} < \infty$. [2]

8. Let $f, Qf \varepsilon L'(r)$ where $(Qf)(x) = xf(x)$. Show that $\widehat{f}$, is a differentiable function. Find a relation between derivative of $\widehat{f}$ and $\widehat{(Qf)}$ and prove your claim. [3]

9. Let $f \varepsilon L'(R)$ and continuous. Show that

   lim inf $|f(t)| = 0$

   $|t| \longrightarrow \infty$ [3]

10. Let $f, f' \varepsilon L'(R)$. Find a relation between $Q\widehat{f}$ and $\widehat{f'}$ and prove your claim. [2]

11. Let $B_0 = \chi_{[0,1]}$, the indicator function of the interval $[0, 1]$. Show that $B_n = B_0 * B_o * \cdots * B_0$, convolution of $B_0, n+1$ times, is in $C^{n-1}(R)$ and B_n is a polynomial of degree $\leq n$ on each interval $[j, j+1]$ for j any integer. [3]

12. Let $f \varepsilon L'(R)$ with $\int f(t)dt = 0$ and *supp* $f \subset [-a, a]$ for some $a > 0$. Let H be the Hilbert transform of f. show that $\int\limits_{|t| \geq 2a} |H(f)(t)|\ dt \leq$

    $K\|f\|_1$.

    for some constant K independent of f. [3]

13. Let $f \varepsilon L'(R), f \geq 0$, and $\lambda > 0$. State and prove Calderon Zygmund decomposition for f at level λ. [5]

14. Let $f \varepsilon C'[0, 2\pi]$ with $f(0) = f(2\pi)$. Let $S_n(f)$ be the fourier series associated with f. Show that $\|S_n(f) - f\|_\infty \longrightarrow 0$ as $n \longrightarrow \infty$. [3]

15. Let $f \varepsilon L'(R)$ with bounded support. State Paley-Wiener theorem for $\widehat{f}$. [2]